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Wigner Functions for Klein-Gordon Oscillators in Non-commutative Space

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Abstract As a quasi-probability distribution function in phase-space and as well as a special representation of the density matrix, the Wigner function is of great significance in Physics. This letter first makes a review of Wigner function and then provides three approaches of calculating it in non-commutative space. Finally, with the help of Moyal-Weyl multiplication and Bopp's shift, the Wigner functions for Klein-Gordon oscillators in non-commutative space are deduced explicitly.

Keywords Wigner function · Klein-Gordon oscillator · Non-commutative space

1 Introduction

The Wigner function was first introduced by Wigner in 1932 [1], recently it has enjoyed a wide popularity in virtually all areas of physics. In fact, as a quasi-probability distribution function in phase-space as well as a special representation of the density matrix, it is of great value in quantum measurement. And it has been useful in describing quantum transport in quantum optics, nuclear physics, decoherence, quantum computing, quantum chaos, signal processing, etc. Nevertheless, a remarkable aspect [2, 3] of the Wigner function was not pioneered until 1975 by Moyal, where Wigner function was proved to satisfy an Moyal $*_\hbar$-eigenvalue equation. Besides the Schrodinger and Heisenberge operator quantization method and Feynman's path integral quantization, the Wigner function is

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an another formulation of quantum mechanics. In this logically complete and self-standing formulation, one needs not choose sides-coordinate or momentum space because the function works in full phase-space, accommodating some uncertainty principles. What's more, as a time-independent function and a quasi-probability distribution function in phase space, the Wigner function is of significance in modern quantum measurement. Take [3] for example, the Wigner function of an ensemble of helium atoms was skillfully tested and the result is as what of the theoretical calculation.

After the Moyal-quantization was made, recently the concept of non-commutative geometry [4] was introduced by string theorists in the string scale. That is, in the ultra-micro-areas (i.e. Planck scale), the space-time coordinates are not commutative. Thus, the concept of space-time point is meaningless as a result of the uncertainty among space-time coordinates. Therefore, it is necessary to have a new space-time geometry, and the new physical effects caused by the non-commutativity among spaces should be studied. In Refs. [5–7], hydrogen atomic energy level and its Lamb shift are calculated under the framework of non-commutative quantum electrodynamics theory; the results show that deviation exist between the non-commutative quantum electrodynamics and ordinary electrodynamics in classical electrodynamics and quantum level, and the deviation depends on the space and space non-commutative parameters. References [8–12] give algebraic expression of coordinates and momentum in non-commutative phase space, as an example, Refs. [8–12] makes a study of arbitrary-dimensional harmonic oscillators in non-commutative space, and finally obtains the mapping of Schrodinger equation from non-commutative phase space to commutative phase space. Moreover, in recent years gratifying achievements have been made in the topological phase in non-commutative quantum Mechanics [13–17] and non-commutative amendment of energy level has also been studied in [18–29].

The present paper is devoted to study the non-commutative properties of Wigner function for Klein-Gordon oscillators. The paper is organized as follows: in Sect. 2, we discuss the wave functions for Klein-Gordon oscillators; in Sect. 3, we give Wigner functions for Klein-Gordon oscillators in commutative space; in Sect. 4, we introduce three methods of calculating Wigner functions in non-commutative space, and by using the first approach, the Wigner functions for Klein-Gordon oscillators in non-commutative space is obtained. Conclusions are given in the last section.

2 Wave Equations and Wave Functions of Klein-Gordon Oscillators in Communicative Space

The Klein-Gordon oscillators in commutative space can be described by the following equation [26]

$$c^2(\vec{p}+im\omega\vec{r})\cdot(\vec{p}-im\omega\vec{r})\psi=(E^2-m^2c^4)\psi. \tag{1}$$

In the two-dimensional space the equation above can also be rewritten as

$$c^2[p_1^2+p_2^2+m^2\omega^2(x_1^2+x_2^2)]\psi=(E^2-m^2c^4+2mc^2\hbar\omega)\psi. \tag{2}$$

Set

$$H=p_1^2+p_2^2+m^2\omega^2(x_1^2+x_2^2),\qquad \tilde{E}=(E^2-m^2c^4+2mc^2\hbar\omega)/c^2. \tag{3}$$

Then (2) changes to

$$H\psi = \tilde{E}\psi. \tag{4}$$

This is the eigenvalue equation of a Klein-Gordon oscillator in commutative space. Thus the Klein-Gordon oscillators' eigenfunctions and energy eigenvalues are obtained as follows,

$$\psi_{n_1 n_2}(x_1, x_2) = N_{n_1} N_{n_1} e^{-\frac{(\varepsilon^2 x_1^2 + \varepsilon^2 x_2^2)}{2}} H_{n_1}(\varepsilon x_1) H_{n_1}(\varepsilon x_2), \tag{5}$$

$$E^2_{n_1 n_2} = 2mc^2\hbar\omega(n_1 + n_2 + 1) + m^2c^4 - 2mc^2\hbar\omega, \tag{6}$$

where $N_n = (\frac{\varepsilon}{n!2^n\sqrt{\pi}})^{\frac{1}{2}}, \varepsilon = \sqrt{m\omega/\hbar}$ and H_n is the Hermite polynomials, $n_1, n_2 = 0, 1, 2, 3, \ldots$.

3 Wigner Functions of Klein-Gordon Oscillators in Commutative Space

In this section we mainly discuss Wigner functions of Klein-Gordon oscillators in commutative space. As is known that there are three logical self-consistent methods of quantization from classical mechanics to quantum mechanics. The first one is operators regularization in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the twenties of last century. The second one is Path Integrals conceived by Dirac and constructed by Feynman. The third one is the Moyal $*_\hbar$-product quantization based on Wigner's (1932) quasi-probability distribution function. In this quantization Wigner function's energy eigenvalue equation is described by the following star eigenvalue equation

$$H(x, p) *_\hbar W_n(x, p) = W_n(x, p) *_\hbar H(x, p) = E_n W_n(x, p), \tag{7}$$

where the $*_\hbar$-product is defined as

$$*_\hbar \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}. \tag{8}$$

It is known that in phase space with the n degree of freedom, the form of the Wigner function given by Wigner himself is

$$W(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} d\vec{y} e^{-i\vec{y}\cdot\vec{p}} \psi^*\left(\vec{x} - \frac{\hbar}{2}\vec{y}\right) \psi\left(\vec{x} + \frac{\hbar}{2}\vec{y}\right), \tag{9}$$

where $\psi(\vec{x})$ is the static eigenfunction of Schrodinger equation. In the two-dimensional case, the Wigner function is rewritten as

$$W(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{y} e^{-i\vec{y}\cdot\vec{p}} \psi^*\left(\vec{x} - \frac{\hbar}{2}\vec{y}\right) \psi\left(\vec{x} + \frac{\hbar}{2}\vec{y}\right). \tag{10}$$

This is a special representation of the density matrix. Alternatively, it is an autocorrelation function of a wave function $\psi(x)$ in a quantum system.

After inserting (5) into (10), the Wigner function of Klein-Gordon oscillators in commutative space can be written as,

$$W_{n_1n_2}(\vec{x},\vec{p}) = \frac{N_{n_1}^2 N_{n_2}^2}{(2\pi)^2}\int_{-\infty}^{\infty} d\vec{y}\exp\biggl[-\frac{(x_1-\frac{\hbar}{2}y_1)^2+(x_2-\frac{\hbar}{2}y_2)^2}{2\hbar} - \frac{(x_1+\frac{\hbar}{2}y_1)^2+(x_2+\frac{\hbar}{2}y_2)^2}{2\hbar}\biggr] \times H_{n_1}\biggl[\sqrt{\frac{1}{\hbar}}\biggl(\vec{x}-\frac{\hbar}{2}\vec{y}\biggr)\biggr]H_{n_2}\biggl[\sqrt{\frac{1}{\hbar}}\biggl(\vec{x}+\frac{\hbar}{2}\vec{y}\biggr)\biggr]e^{-i\vec{y}\vec{p}}, \tag{11}$$

where, for simplicity, the $m=\omega=1$ has been chosen. Since (11) has the symmetry on variable x_1, p_1 and x_2, p_2, and it can been factorized as $W_{n_1n_2}(\vec{x},\vec{p}) = W_{n_1}(x_1,p_1)W_{n_2}(x_2,p_2)$, and it is not very difficult to verify that,

$$W_{n_1}(x_1,p_1) = \frac{N_{n_1}^2}{2\pi}\int_{-\infty}^{\infty} dy_1\exp\biggl[-iy_1p_1 - \frac{(x_1-\frac{\hbar}{2}y_1)^2}{2\hbar} - \frac{(x_1+\frac{\hbar}{2}y_1)^2}{2\hbar}\biggr] \times H_{n_1}\biggl[\sqrt{\frac{1}{\hbar}}\biggl(x_1-\frac{\hbar}{2}y_1\biggr)\biggr]H_{n_1}\biggl[\sqrt{\frac{1}{\hbar}}\biggl(x_1+\frac{\hbar}{2}y_1\biggr)\biggr]. \tag{12}$$

Inserting Hermite polynomial's generating function into (12), we have

$$W_{n_1} = \frac{N_{n_1}^2}{2\pi\sqrt{\hbar}}e^{-x_1^2/\hbar}\biggl\{\frac{\partial^{2n_1}}{\partial s^{n_1}\partial t^{n_1}}\exp\biggl[\sqrt{\frac{4}{\hbar}}x_1(s+t) - 2st - i\sqrt{\frac{4}{\hbar}}p_1(t-s)\biggr] \times \int_{-\infty}^{\infty} d\nu e^{-\frac{1}{4}\nu^2+\frac{ip_1\nu}{\sqrt{\hbar}}}\biggr\}\biggr|_{s=0,t=0} \tag{13}$$

where $\nu = \sqrt{\hbar}y_1 + 2(t-s)$. With the integral formula

$$\int_{-\infty}^{\infty} dx\exp[-\beta^2x^2 \pm 2i\gamma x] = \frac{\sqrt{\pi}}{\beta}\exp\biggl[-\frac{\gamma^2}{\beta^2}\biggr], \tag{14}$$

equation (13) changes to

$$W_{n_1}(x_1,p_1) = \frac{N_{n_1}^2}{\sqrt{\pi\hbar}}e^{-(x_1^2+p_1^2)/\hbar}\biggl\{\frac{\partial^{2n_1}}{\partial s^{n_1}\partial t^{n_1}} \times \exp\biggl[\sqrt{\frac{4}{\hbar}}x_1(s+t) - 2st - i\sqrt{\frac{4}{\hbar}}p_1(t-s)\biggr]\biggr\}\biggr|_{s=0,t=0}. \tag{15}$$

If we write,

$$Y_{n_1} = \frac{\partial^{2n_1}}{\partial s^{n_1}\partial t^{n_1}}\exp\biggl[\sqrt{\frac{4}{\hbar}}x_1(s+t) - 2st - i\sqrt{\frac{4}{\hbar}}p_1(t-s)\biggr], \tag{16}$$

then we obtain

$$W_{n_1}(x_1, p_1) = \frac{N_{n_1}^2}{\sqrt{\pi\hbar}} e^{-(x_1^2+p_1^2)/\hbar} Y_{n_1}|_{s=0,t=0}. \tag{17}$$

After some calculations, we find

$$\begin{aligned} Y_{n_1} &= \sum_{k=0}^{n_1} \frac{(-1)^{n_1+k}(n_1!)^3 2^{n_1-k}}{(k!)^2(n_1-k)!} \left(\frac{2x_1}{\sqrt{\hbar}} - 2t + \frac{2ip_1}{\sqrt{\hbar}}\right)^k \left(\frac{2x_1}{\sqrt{\hbar}} - 2s - \frac{2ip_1}{\sqrt{\hbar}}\right)^k \\ &\quad \times \exp\left[2x_1(s+t) - 2ts - \frac{2ip_1(t-s)}{\hbar}\right]. \end{aligned} \tag{18}$$

Inserting (18) into (17), we have

$$W_{n_1}(x_1, p_1) = \frac{(-1)^{n_1}}{\pi\hbar} e^{-(x_1^2+p_1^2)/\hbar} \sum_{k=0}^{n_1} \frac{(-1)^k (n_1!)^2}{(k!)^2(n_1-k)!} \left[\frac{2}{\hbar}(x_1^2+p_1^2)\right]^k. \tag{19}$$

If we set $\xi = 2(x_1^2 + p_1^2)/\hbar$, then (19) can be rewritten as

$$W_{n_1}(x_1, p_1) = \frac{(-1)^{n_1}}{\pi\hbar} e^{-\frac{\xi}{2}} L_{n_1}(\xi), \tag{20}$$

where

$$L_{n_1}(\xi) = \sum_{k=0}^{n} \frac{(-1)^k (n_1!)^2}{(k!)^2(n_1-k)!} \xi^k = \frac{1}{n_1!} e^{\xi} \frac{\partial^{n_1}}{\partial \xi^{n_1}} (e^{-\xi} \xi^{n_1}) \tag{21}$$

is the Laguerre's polynomials. Thus, according to the symmetry the following explicit expression for Wigner function is in order,

$$W_{n_1 n_2} = \frac{(-1)^{n_1+n_2}}{(\pi\hbar)^2} e^{-(\xi+\eta)/2} L_{n_1}(\xi) L_{n_2}(\eta), \tag{22}$$

where $\eta = 2(x_2^2 + p_2^2)/\hbar$. In terms of variables in phase space, (22) is rewritten as

$$\begin{aligned} W_{n_1 n_2}(x_1, p_1, x_2, p_2) &= \frac{(-1)^{n_1+n_2}}{(\pi\hbar)^2} e^{-(x_1^2+p_1^2+x_2^2+p_2^2)/\hbar} \\ &\quad \times L_{n_1}\left[\frac{2}{\hbar}(x_1^2+p_1^2)\right] L_{n_2}\left[\frac{2}{\hbar}(x_2^2+p_2^2)\right]. \end{aligned} \tag{23}$$

This is the very Wigner function of Klein-Gordon oscillators in commutative space. While, in a ground state, (23) changes to

$$W_{00}(x_1, p_1, x_2, p_2) = \frac{1}{(\pi\hbar)^2} e^{-(x_1^2+p_1^2+x_2^2+p_2^2)/\hbar}. \tag{24}$$

This is Gaussian function in phase space which is of great significance in the physical measurements.

4 The Wigner Functions in Non-commutative Space

Now we are in the position to discuss how to calculate Wigner functions in non-commutative (NC) space. As is known that in NC space the coordinate $\hat{x}_i$ and momentum $\hat{p}_i$ operators satisfy the following commutation relations

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \tag{25}$$

where (Θ_{ij}), a constant anti-symmetric tensor composed by the NC parameter θ, is written as the following in the two-dimensional space

$$(\Theta_{ij}) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}. \tag{26}$$

By replacing the normal product with a star product, the Schrödinger equation in commuting space will change into the Schrödinger equation in NC space, i.e. in NC space, the Schrödinger equation can be written as

$$H(p, x) *_\theta \hat{\psi}(x) = E\hat{\psi}(x), \tag{27}$$

where the Moyal-Weyl (or star) product is defined as

$$(f *_\theta g)(x) = e^{\frac{i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = f(x)g(x) + \frac{i}{2}\Theta_{ij}\partial_i f \partial_j g\big|_{x_i=x_j} + \mathcal{O}(\theta^2). \tag{28}$$

Here $f(x)$ and $g(x)$ are two arbitrary functions. Instead of solving the NC Schrödinger equation by using the star product procedure, we use Bopp's shift

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij} p_j, \quad \hat{p}_i = p_i. \tag{29}$$

Thus, the Schrödinger equation becomes

$$\hat{H}(p, x) *_\theta \hat{\psi}(x) = \hat{H}(\hat{p}, \hat{x})\hat{\psi}(x) = E\hat{\psi}(x), \tag{30}$$

in the two-dimensional situation

$$\begin{aligned} \hat{x}_1 &= x_1 - \frac{1}{2\hbar}\theta p_2, \qquad & \hat{x}_2 &= x_2 + \frac{1}{2\hbar}\theta p_1, \\ \hat{p}_1 &= p_1, \qquad & \hat{p}_2 &= p_2. \end{aligned} \tag{31}$$

Here, the NC parameter θ is a very small constant $\leq (10^4 \text{ GeV})^{-2}$ [5–7].

There are three approaches to calculate the Wigner functions in non-commutative space. The first one is to replace the general product in commutative space with the $*_\hbar$-product [30, 31]. Thus, (10) becomes

$$\hat{W}(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\vec{y}\, e^{-i\vec{y}\cdot\vec{p}}\psi^*\left(\vec{x} - \frac{\hbar}{2}\vec{y}\right) *_\theta \psi\left(\vec{x} + \frac{\hbar}{2}\vec{y}\right). \tag{32}$$

In this approach, the integral form of $*_\theta$-product of two functions f and g is useful, and it reads,

$$f(x_1,x_2)*_\theta g(x_1,x_2)=\frac{1}{(\pi\theta)^2}\int_{-\infty}^{\infty}dx_1'dx_2'dx_1''dx_2''\exp\left[\frac{2i}{\theta}\det\begin{pmatrix}1&1&1\\x_1&x_1'&x_1''\\x_2&x_2'&x_2''\end{pmatrix}\right]. \quad (33)$$

The second approach is to calculate $\hat{\psi}$'s star eigenvalue equation (30). With $\hat{\psi}$, we can calculate the integral of $\hat{\psi}$ directly. In this way, the Wigner function in Non-commutative space can be written as

$$\hat{W}(\vec{x},\vec{p})=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\vec{y}\,e^{-i\vec{y}\cdot\vec{p}}\;\hat{\psi}^*\left(\vec{x}-\frac{\hbar}{2}\vec{y}\right)\hat{\psi}\left(\vec{x}+\frac{\hbar}{2}\vec{y}\right), \quad (34)$$

while, because of non-commutative items it is more complicated to calculate this equation than to deal with Schrodinger equation in commutative space.

The third one is to calculate the following WF's $*$-genvalue equations [32]

$$H(\vec{x},\vec{p})*\hat{W}(\vec{x},\vec{p})=E\hat{W}(\vec{x},\vec{p}) \quad (35)$$

and

$$\hat{W}(\vec{x},\vec{p})*H(\vec{x},\vec{p})=E\hat{W}(\vec{x},\vec{p}), \quad (36)$$

in which

$$*=*_\hbar *_\theta=\exp\left(\frac{i\hbar}{2}\sum_{i=1}^{2}(\overleftarrow{\partial}_{x_i}\overrightarrow{\partial}_{p_i}-\overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{x_i})+\frac{i\theta}{2}(\overleftarrow{\partial}_{x_1}\overrightarrow{\partial}_{x_2}-\overleftarrow{\partial}_{x_2}\overrightarrow{\partial}_{x_1})\right). \quad (37)$$

Using Bopp's shift and ordering $H(\vec{x},\vec{p})\to\hat{H}(\hat{\vec{x}},\hat{\vec{p}})$, we have

$$H(\vec{x},\vec{p})*\hat{W}(\vec{x},\vec{p})=\hat{H}(\hat{\vec{x}},\hat{\vec{p}})*_\hbar\hat{W}(\vec{x},\vec{p})=E\hat{W}(\vec{x},\vec{p}), \quad (38)$$

and

$$\hat{W}(\vec{x},\vec{p})*H(\vec{x},\vec{p})=\hat{W}(\vec{x},\vec{p})*_\hbar\hat{H}(\hat{\vec{x}},\hat{\vec{p}})=E\hat{W}(\vec{x},\vec{p}). \quad (39)$$

Now, we use the first approach to calculate the Wigner function for Klein-Gordon oscillators in NC space. Inserting (5) into (32) and in terms of (33), we can have

$$\begin{aligned}\hat{W}_{n_1n_2}(\hat{\vec{x}},\hat{\vec{p}})=&\frac{N_{n_1}^2N_{n_2}^2}{(2\pi)^2(\pi\theta)^2}\int_{-\infty}^{\infty}d\vec{y}dx_1'dx_2'dx_1''dx_2''\,e^{-i\vec{y}\vec{p}}\exp\left[\frac{2i}{\theta}\det\begin{pmatrix}1&1&1\\x_1&x_1'&x_1''\\x_2&x_2'&x_2''\end{pmatrix}\right]\\&\times H_{n_1}\left[\sqrt{\frac{1}{\hbar}}\left(x_1'-\frac{\hbar}{2}y_1\right)\right]H_{n_1}\left[\sqrt{\frac{1}{\hbar}}\left(x_1''+\frac{\hbar}{2}y_1\right)\right]\\&\times H_{n_2}\left[\sqrt{\frac{1}{\hbar}}\left(x_2'-\frac{\hbar}{2}y_2\right)\right]H_{n_2}\left[\sqrt{\frac{1}{\hbar}}\left(x_2''+\frac{\hbar}{2}y_2\right)\right]\\&\times\exp\left[\frac{(x_1-\frac{\hbar}{2}y_1)^2+(x_2-\frac{\hbar}{2}y_2)^2}{2\hbar}+\frac{(x_1+\frac{\hbar}{2}y_1)^2+(x_2+\frac{\hbar}{2}y_2)^2}{2\hbar}\right].\end{aligned} \quad (40)$$

Calculating the integral form of variables $\vec{y}$, we can obtain

$$\hat{W}_{n_1 n_2}(\vec{\hat{x}}, \vec{\hat{p}}) = \frac{4\times(-1)^{m+n}}{(\pi\theta)^2(\pi\hbar)^2}\int_{-\infty}^{\infty} d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 \\ \times \exp\left[\frac{4i\beta_1}{\theta}(\alpha_2 - x_2) - \frac{4i\beta_1}{\theta}(\alpha_1 - x_1)\right] \\ \times \exp\left[-\frac{\alpha_1^2+\alpha_2^2+p_1^2+p_2^2}{\hbar} - \frac{2ip_1\beta_1}{\hbar} + \frac{2ip_2\beta_2}{\hbar}\right] \\ \times L_{n_1}\left[\frac{2}{\hbar}(\alpha_1^2+p_1^2)\right] L_{n_2}\left[\frac{2}{\hbar}(\alpha_2^2+p_2^2)\right], \tag{41}$$

where

$$\alpha_1 = \frac{1}{2}(x_1' + x_1''), \qquad \beta_1 = \frac{1}{2}(x_1' - x_1''), \qquad \alpha_2 = \frac{1}{2}(x_2' + x_2''), \qquad \beta_2 = \frac{1}{2}(x_2' - x_2''). \tag{42}$$

Using the integral formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ikx} = \delta(k), \tag{43}$$

and calculating the integral of variables β_1 and β_2, we have

$$\hat{W}_{n_1 n_2}(\vec{\hat{x}}, \vec{\hat{p}}) = \frac{(-1)^{n_1+n_2}}{(\pi\hbar)^2}\int_{-\infty}^{\infty} d\alpha_1 d\alpha_2 \delta\left(\alpha_1 - x_1 + \frac{\theta}{2\hbar}p_2\right)\delta\left(\alpha_2 - x_2 - \frac{\theta}{2\hbar}p_1\right) \\ \times \exp\left[-\frac{\alpha_1^2+\alpha_2^2+p_1^2+p_2^2}{\hbar}\right] L_{n_1}\left[\frac{2}{\hbar}(\alpha_1^2+p_1^2)\right] L_{n_2}\left[\frac{2}{\hbar}(\alpha_2^2+p_2^2)\right]. \tag{44}$$

Thus, we obtain

$$\hat{W}_{n_1 n_2}(x_1, p_1, x_2, p_2) = \frac{(-1)^{n_1+n_2}}{(\pi\hbar)^2}\exp\left[-\frac{x_1^2+x_2^2+p_1^2+p_2^2}{\hbar}\right. \\ \left. - \frac{\theta}{\hbar^2}(x_1p_2 - x_2p_1) + \frac{\theta^2}{4\hbar^3}(p_1^2+p_2^2)\right] \\ \times L_{n_1}\left[\frac{2}{\hbar}\left(x_1^2+p_1^2 - \frac{\theta}{\hbar}x_1p_2 + \frac{\theta^2}{4\hbar^2}p_2^2\right)\right] \\ \times L_{n_2}\left[\frac{2}{\hbar}\left(x_2^2+p_2^2 - \frac{\theta}{\hbar}x_2p_1 + \frac{\theta^2}{4\hbar^2}p_1^2\right)\right]. \tag{45}$$

Neglecting small amounts of high-end included θ^2, we have

$$\hat{W}_{n_1 n_2}(x_1, p_1, x_2, p_2) = \frac{(-1)^{n_1+n_2}}{(\pi\hbar)^2} \\ \times \exp\left[-\frac{x_1^2+x_2^2+p_1^2+p_2^2}{\hbar} - \frac{\theta}{\hbar^2}(x_1p_2 - x_2p_1)\right] \\ \times L_{n_1}\left[\frac{2}{\hbar}\left(x_1^2+p_1^2 - \frac{\theta}{\hbar}x_1p_2\right)\right] L_{n_2}\left[\frac{2}{\hbar}\left(x_2^2+p_2^2 - \frac{\theta}{\hbar}x_2p_1\right)\right]. \tag{46}$$

This is the Wigner function of Klein-Gordon oscillators in non-commutative space. While, in a ground state the Wigner function reads

$$\hat{W}_{00}(x_1,p_1,x_2,p_2)=\frac{1}{(\pi\hbar)^2}\exp\left[-\frac{x_1^2+x_2^2+p_1^2+p_2^2}{\hbar}-\frac{\theta}{\hbar^2}(x_1p_2-x_2p_1)\right]. \tag{47}$$

Once the non-commutative parameter $\theta=0$, (46)–(47) return back as that in commutative space.

5 Conclusion Remarks

In brief, in order to study Wigner functions for Klein-Gordon oscillators in non-commutative space reasonably, this paper first provides the Wigner function for Klein-Gordon oscillators in commutative space. Then, after explaining three approaches of calculating Wigner functions, instead of doing tedious calculation, the Wigner functions for Klein-Gordon oscillators are calculated by integral method in non-commutative space. Obviously, if the non-commutative parameter θ=0, the Wigner function of Klein-Gordon oscillators in non-commutative space is to be that in commutative space. This effect in non-commutative space is expected to be tested at a very high energy level, and the experimental observation of the effect remains to be further studied.

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